SA402 Fall 2023 · Final Exam · List of Formulas

Markov chains

- n-step transition probabilities: $\mathbf{P}^{(n)} = \mathbf{P}^n$
- n-step state probabilities: $\mathbf{q}^{(n)\top} = \mathbf{q}^\top \mathbf{P}^n$
- First-passage probabilities, starting in $\mathcal{A}$ and ending in $\mathcal{B}$ in the nth step: $\mathbf{F}^{(n)}_{\mathcal{AB}} = \mathbf{P}^{n-1}_{\mathcal{AA}} \mathbf{P}_{\mathcal{AB}}$
- Steady-state probabilities for recurrent class $\mathcal{R}$:

$$\pi^\top_{\mathcal{R}} \mathbf{P}_{\mathcal{RR}} = \pi^\top_{\mathcal{R}}$$
$$\pi^\top_{\mathcal{R}} \mathbf{1} = 1$$

- Absorption probabilities for transient states $\mathcal{T}$ and absorbing state $\mathcal{R} = \{j\}$:

$$\alpha_{\mathcal{TR}} = \mathbf{N}\mathbf{P}_{\mathcal{TR}} \quad \text{where} \quad \mathbf{N} = (\mathbf{I} - \mathbf{P}_{\mathcal{TT}})^{-1}$$

- Expected time to absorption for transient states $\mathcal{T}$: $\mu_{\mathcal{T}} = \mathbf{N1}$

Poisson processes

Useful distributions:

	$X \sim \text{Poisson}(\mu)$	$X \sim \text{Exponential}(\lambda)$	$X \sim \text{Erlang}(n, \lambda)$
pmf / pdf	$p_X(a) = \begin{cases} \dfrac{e^{-\mu}\mu^a}{a!} & \text{if } a = 0, 1, 2, \ldots \\ 0 & \text{o/w} \end{cases}$	$f_X(a) = \begin{cases} \lambda e^{-\lambda a} & \text{if } a \geq 0 \\ 0 & \text{o/w} \end{cases}$	$f_X(a) = \begin{cases} \dfrac{\lambda(\lambda a)^{n-1} e^{-\lambda a}}{(n-1)!} & \text{if } a \geq 0 \\ 0 & \text{o/w} \end{cases}$
cdf	$F_X(a) = \displaystyle\sum_{k=0}^{\lfloor a \rfloor} \frac{e^{-\mu}\mu^k}{k!}$	$F_X(a) = \begin{cases} 1 - e^{-\lambda a} & \text{if } a \geq 0 \\ 0 & \text{o/w} \end{cases}$	$F_X(a) = \begin{cases} 1 - \displaystyle\sum_{k=0}^{n-1} \frac{e^{-\lambda a}(\lambda a)^k}{k!} & \text{if } a \geq 0 \\ 0 & \text{o/w} \end{cases}$
expected value	$E[X] = \mu$	$E[X] = \dfrac{1}{\lambda}$	$E[X] = \dfrac{n}{\lambda}$
variance	$\text{Var}(x) = \mu$	$\text{Var}(X) = \dfrac{1}{\lambda^2}$	$\text{Var}(X) = \dfrac{n}{\lambda^2}$

Markov processes

- Steady-state probabilities:

$$\pi^\top \mathbf{G} = \mathbf{0}^\top$$
$$\pi^\top \mathbf{1} = 1$$

Birth-death processes

- Steady-state probabilities:

$$d_0 = 1 \qquad d_j = \frac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{\mu_1 \mu_2 \cdots \mu_j} \quad \text{for } j = 1, 2, \ldots \qquad D = \sum_{i=0}^{\infty} d_i \qquad \pi_j = \frac{d_j}{D} \quad \text{for } j = 0, 1, 2, \ldots$$

- Expected number of customers in the system: $\ell = \sum_{n=0}^{\infty} n \pi_n$
- Expected number of customers in the queue, s parallel servers: $\ell_q = \sum_{n=s+1}^{\infty} (n - s) \pi_n$
- Effective arrival rate: $\lambda_{\text{eff}} = \sum_{i=0}^{\infty} \lambda_i \pi_i$
- Little's law (system-wide): $\ell = \lambda_{\text{eff}} w$
- Little's law (queue only): $\ell_q = \lambda_{\text{eff}} w_q$

Standard queueing models

M/M/∞:

- Steady-state probabilities: $\pi_j = \Pr\{L = j\}$ where $L \sim \text{Poisson}(\lambda/\mu)$

M/M/s:

- Steady-state probabilities:

$$\rho = \frac{\lambda}{s\mu} \qquad \pi_0 = \left[\left(\sum_{j=0}^{s} \frac{(s\rho)^j}{j!} \right) + \frac{s^s \rho^{s+1}}{s!(1-\rho)} \right]^{-1} \qquad \pi_j = \begin{cases} \dfrac{(\lambda/\mu)^j}{j!} \pi_0 & \text{for } j = 1, 2, \ldots, s \\ \dfrac{(\lambda/\mu)^j}{s! s^{j-s}} \pi_0 & \text{for } j = s+1, s+2, \ldots \end{cases}$$

- Expected number of customers in queue: $\ell_q = \dfrac{\pi_s \rho}{(1-\rho)^2}$
- Expected number of customers in the system: $\ell = \ell_q + \dfrac{\lambda}{\mu}$

G/G/s:

- Whitt's approximation:

$$G = \text{generic interarrival time random variable with rate } \lambda = \frac{1}{E[G]}$$

$$X = \text{generic service time random variable with rate } \mu = \frac{1}{E[X]}$$

$$\varepsilon_G = \frac{\text{Var}[G]}{E[G]^2} \qquad \varepsilon_X = \frac{\text{Var}[X]}{E[X]^2} \qquad \hat{w}_q \approx \frac{\varepsilon_G + \varepsilon_X}{2} w_q$$